Supersymmetry on Curved Spaces

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Outline

- Introduction and Motivations
- Supersymmetry on Curved Backgrounds
- The Euclidean case
- The Lorentzian case
- Conclusions and Open Problems

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some related results in
T. Dumitrescu, G. Festuccia, N. Seiberg arXiv:1205.1115

Supersymmetry

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- String theory strongly relies on the existence of extra-dimensions. The requirement that a vacuum is supersymmetric selects particular type of internal geometries (Calabi-Yau, Generalized Geometries, ...) which have been objects of intense investigation in the last thirty years.
- More recent interest in an even simpler question: when can we define a supersymmetric quantum field theory on a nontrivial manifold M ? Familiar examples: $\mathbb{R}^{p,q}$, AdS_d, $\mathbb{R}^{p,q} \times T^s$,... More general analysis just started ...

Festuccia, Seiberg
Santleben, Tsimpis
Klare, Tomasiello, A. Z.
Dumitrescu, Festuccia, Seiberg
Cassani, Klare, Martelli, Tomasiello, A. Z.
Liu, Pando Zayas, Reichmann; de Medeiros
Dumitrescu, Festuccia; Kehagias-Russo [...]

Why should we be interested in susy in curved space?

- Mathematically: 20 years ago Witten showed on to put an $N = 2$ supersymmetric theory on any Euclidean manifold M. The result is a topological theory (links to Donaldson invariants of M ,....)

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- Physically: more recently we learned how to compute the full quantum partition function of Euclidean supersymmetric theories on spheres and other manifolds, reducing it to a matrix model.

$$\int \prod_{i=1}^{N_1} du_i \prod_{j=1}^{N_2} dv_j \frac{\prod_{i<j} \sinh^2 \frac{u_i - u_j}{2} \sinh^2 \frac{v_i - v_j}{2}}{\prod_{i<j} \cosh^2 \frac{u_i - v_j}{2}} e^{\frac{ik}{4\pi}\left(\sum u_i^2 - \sum v_j^2\right)}$$

ABJM, 3d Chern-Simon theories, [Kapustin,Willet,Yakoov;Drukker,Marino,Putrov]

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- Mathematically: 20 years ago Witten showed how to put an $N = 2$ supersymmetric theory on any Euclidean manifold M. The result is a topological theory which computes Donaldson invariants of M

- Physically: more recently we learned how to compute the full quantum partition function of Euclidean supersymmetric theories on spheres and other manifolds, reducing it to a matrix model.

- Holographically: we recently found new examples of regular asymptotically AdS backgrounds dual to CFTs on curved space-times [Martelli,Passias,Sparks,....]

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r))$$

Supersymmetric theories

Supersymmetric theories are usually formulated on Minkowski space-time $\mathbb{R}^{3,1}$. At the classical level, we have an action for bosonic and fermionic fields

$$S_{\mathrm{SUSY}}(\phi(x), \psi(x), A_\mu(x), ...)$$

invariant under transformations that send bosons into fermions and viceversa

$$\delta\phi(x) = \epsilon\psi(x)\,, \qquad \delta\psi = \partial_\mu\phi\gamma^\mu\epsilon + \cdots$$

where ϵ is a constant spinor.

The symmetry group of the theory contains translations, Lorentz transformations $SO(3,1)$ and the fermonic symmetries with the corresponding fermionic Noether charges Q. The theory can be also formulated on Euclidean space $\mathbb{R}^4$.

Can we define the theory on a general manifold M preserving supersymmetry?

Supersymmetric theories on curved spaces

The general strategy is to promote the metric to a dynamical field [Festuccia,Seiberg] .

This is done by coupling the rigid theory to the multiplet of supergravity $(g_{\mu\nu}, \psi_\mu, ...)$

$$S_{\rm SUGRA}(\phi(x), \psi(x), g_{\mu\nu}(x), \psi_\mu(x), ...)$$

which is invariant under local transformations

$$\delta\phi(x) = \epsilon(x)\psi(x)\,, \qquad \delta e^a_\mu(x) = \bar{\epsilon}(x)\gamma^a\psi_\mu(x) + \cdots$$

We are gauging the original symmetries of the theory. At linear level this is just the Noether coupling

$$-\frac{1}{2} g_{mn} T^{mn} + \bar{\psi}_m \mathcal{J}^m$$

Supersymmetric theories on curved spaces

The rigid theory is obtained by freezing the fields of the metric multiplet to background values

$$g_{\mu\nu} = g^M_{\mu\nu}\,, \quad \psi_\mu = 0$$

The resulting theory will be supersymmetric if the variation of supersymmetry vanish

$$\begin{aligned} \delta e^a_\mu(x) &= \bar{\epsilon}(x)\gamma^a\psi_\mu(x) + \cdots \equiv 0 \\ \delta\psi_\mu(x) &= \nabla_\mu \epsilon + \cdots \equiv 0 \end{aligned}$$

The graviton variation gives a differential equation for $\epsilon(x)$ which need to be solved in order to have supersymmetry and gives constraints on M.

Superconformal theories

It is very interesting physically and mathematically to consider theories that are also invariant under dilatations

$$x \to \lambda x$$

The group of symmetries of a CFT is enlarged to

- translations + Lorentz $SO(3,1) \to$ conformal group $SO(4,2)$
- supersymmetry Q is doubled: (Q, S)
- extra bosonic global symmetries rotating (Q, S) (R-symmetries)
  for $N = 1$ supersymmetry the R-symmetry is $U(1)$: $Q \to e^{i\alpha} Q$.

The possible conformal superalgebras have been classified by Nahm in the seventies. It is $SU(2,2|1)$ for $N = 1$ supersymmetry.

Superconformal theories on curved spaces

The strategy here is to couple the CFT to conformal supergravity. The $N = 1$ conformal supergravity multiplet $(g_{\mu\nu}, \psi_\mu, A_\mu)$ contains gauge fields for the superconformal symmetries

$$-\frac{1}{2} g_{mn} T^{mn} + A_m J^m + \bar{\psi}_m \mathcal{J}^m$$

We freeze $(g_{\mu\nu}, A_\mu)$ to background values and set $\psi_\mu = 0$. In order to preserve some supersymmetry, the gravitino variation must vanish.

$$(\nabla_a - iA_a)\epsilon_+ + \gamma_a \epsilon_- = 0$$

$\epsilon_\pm$ parameters for the supersymmetries and the superconformal transformations.

Superconformal theories on curved spaces

The variation of the gravitino can be written as

$$(\nabla_a - iA_a)\epsilon_+ + \gamma_a \epsilon_- = 0 \qquad \Longrightarrow \qquad \nabla^A_a \epsilon_+ = \frac{1}{d}\gamma_a \not\nabla^A \epsilon_+$$

- ϵ_+ is a charged (or twisted or spin^c) spinor, a section of $\Sigma^+ \otimes \mathcal{L}$
- $\nabla^A_a = \nabla_a - iA_a$ is a connection on $\mathcal{L}$
- $\not\nabla^A \equiv \gamma^a(\nabla_a - iA_a)$ is the twisted Dirac operator

Superconformal theories on curved spaces

The condition for preserving some supersymmetry is then

$$\nabla_a^A \epsilon_+ = \frac{1}{d}\gamma_a \not\nabla^A \epsilon_+$$

- (twisted) conformal Killing equation
- projection of $\nabla_a^A \epsilon$ on the irreducible spin 3/2 component
- conformally invariant equation

Twistors

Conformal Killing Spinors

$$\nabla_a \epsilon = \frac{1}{4} \gamma_a \not{\nabla} \epsilon$$

with $A = 0$ (also called twistors) have been studied and classified:

- Lorentzian: pp-waves and Fefferman metrics.
- Euclidean: conformally equivalent to manifolds with Killing spinors

SCFTs on Euclidean Curved Backgrounds

Conformal Killing Spinors with $A = 0$ are classified: they become Killing Spinors on a Weyl rescaled metric:

$$\nabla_a \epsilon = \frac{1}{d} \gamma_a \not\nabla \epsilon \qquad \Longrightarrow \qquad \nabla_a \epsilon = \mu \gamma_a \epsilon$$

Manifolds with Killing Spinors are in turn classified: in the compact case the cone over it has restricted holonomy:

dim	H	$C(H)$
3	S^3	$\mathbb{R}^4$
4	S^4	$\mathbb{R}^5$
5	Sasaki-Einstein	CY_3
6	Nearly-kähler	G_2 manifolds

or quotients...

SCFTs on Euclidean Curved Backgrounds

Information on spinors ϵ can be encoded in differential forms (bilinears)

$$\Sigma_p \otimes \Sigma_p = \oplus \Lambda^k TM_p$$

$$(\epsilon \otimes \epsilon^\dagger)_{\alpha\beta} = \sum \gamma_{\alpha\beta}^{i_1\cdots i_k} C_{i_1\cdots i_k} , \qquad \Longrightarrow \qquad C_{i_1\cdots i_k} = \epsilon^\dagger \gamma^{i_1\cdots i_k} \epsilon$$

SCFTs on Euclidean Curved Backgrounds

In Euclidean signature from a CKS ϵ_+ we can construct two two forms j, ω:

$$\epsilon_+ \otimes \epsilon_+^\dagger = \frac{1}{4} e^B e^{-ij} \,, \qquad \epsilon_+ \otimes \overline{\epsilon_+} = \frac{1}{4} e^B \omega \,, \qquad e^B \equiv ||\epsilon_+||^2$$

$$j = e_1 \wedge e_2 + e_3 \wedge e_4 \,, \qquad \omega = (e_1 + ie_2) \wedge (e_3 + ie_4)$$

The CKS equation translates into a set of linear constraints for the differential of the bilinears and the gauge field

$$\nabla_a^A \epsilon_+ = \frac{1}{d} \gamma_a \nabla\!\!\!/^A \epsilon_+ \qquad \Longrightarrow \qquad \text{linear eqs for } \{dj, d\omega, A\}$$

SCFTs on Euclidean Curved Backgrounds

The existence of a CKS is (locally) equivalent to the existence of a complex structure.

$$\text{CKS} \qquad \Longrightarrow \qquad \begin{aligned} w^3 &= 0 \qquad \text{(complex manifold)} \\ A_{1,0} &= -i\left(-\frac{1}{2}\overline{w^5_{0,1}} + \frac{1}{4}w^4_{1,0} + \frac{1}{2}\partial B\right) \\ A_{0,1} &= -i\left(+\frac{1}{2}w^5_{0,1} - \frac{3}{4}w^4_{0,1} + \frac{1}{2}\bar\partial B\right) \end{aligned}$$

- $dj = w_4 \wedge j, d\omega = w_5 \wedge \omega + w_3 \wedge \bar\omega$
- Notice that A is in general complex

[klare,tomasiello,A.Z.;dumitruescu,festuccia,seiberg]

Examples: Kähler manifolds

Kähler manifolds are complex and support supersymmetry.

Equivalent characterizations:

- Existence of two-forms with

$$dj = 0\,, \qquad d\omega = 2iA \wedge \omega$$

- existence of a covariantly constant charged spinor

$$(\nabla_m - iA_m)\,\epsilon_+ = 0$$

A in this case is real.

Examples: non Kähler manifolds

The simplest example of a non-Kähler but complex manifold is probably $S^3 \times S^1$.

- it solves the CKS conditions with a complex gauge field: $A = id\phi$
- partition functions on $S^3 \times S^1$ define superconformal indices!

Examples: S^4 revised

Every Killing spinor $\nabla_a \epsilon = \gamma_a \epsilon$ is also Conformal Killing with $A = 0$. However S^4 is not complex!

Solution: j and ω are well defined everywhere only if ϵ_+ never vanishes. The manifold is complex outside zeros of ϵ_+.

Decomposing

$$\epsilon = \epsilon_+ + \epsilon_-$$

$\epsilon_\pm$ vanish at North and South pole respectively: $\mathbb{R}^4 = S^4 - \mathrm{NP}$ is complex.

SCFTs on Lorentzian Curved Backgrounds

In Lorentzian signature from a CKS ϵ_+ we can construct a real null vector z and a complex two form ω:

$$\epsilon_+ \otimes \bar{\epsilon}_+ = z + i * z \ , \qquad \epsilon_+ \otimes \bar{\epsilon}_- \equiv \omega = z \wedge w \, , \qquad z^2 = 0$$

$$z = e_0 - e_1 \, , \qquad w = e_2 + i e_3$$

The CKS equation translates into a set of linear constraints for the differential of the bilinears and the gauge field

$$\nabla^A_a \epsilon_+ = \frac{1}{d} \gamma_a \not{\nabla}^A \epsilon_+ \qquad \Longrightarrow \qquad \text{linear} \quad \text{eqs} \quad \text{for } \{dz, d\omega, A\}$$

SCFTs on Lorentzian Curved Backgrounds

The existence of a charged CKS is equivalent to the existence of a null CKV.

$$\nabla^A_a \epsilon_+ = \frac{1}{4}\gamma_a \not{\nabla}^A \epsilon_+ \qquad \Longrightarrow \qquad \nabla_\mu z_\nu + \nabla_\nu z_\mu = \lambda g_{\mu\nu} \quad (8 \ \ \text{conditions})$$

(12 conditions) real gauge field A (4 conditions)

Whenever there exists a null CKV a SCFT preserves some supersymmetry in curved space. [cassani,klare,martelli, tomasiello,A.Z.]

SCFTs on Lorentzian Curved Backgrounds

Every conformal Killing vector becomes Killing in a Weyl rescaled metric

$$\mathcal{L}_z g_{\mu\nu} = \lambda g_{\mu\nu} \ \Rightarrow \ \mathcal{L}_z(e^{2f} g_{\mu\nu}) = (\lambda + 2z \cdot df) g_{\mu\nu}$$

We can then choose adapted coordinates $z = \partial/\partial y$

$$ds^2 \ = \ 2H^{-1}(du + \beta_m dx^m)(dy + \rho_m dx^m + F(du + \beta_m dx^m)) + Hh_{mn} dx^m dx^n$$

where $H, h_{mn}, \beta_m, \rho_m$ do not depend on y. A is completely determined by these functions.

Summary: the result for $A \neq 0$

Focus on a single solution of the CKS eqs in 4d: $\nabla^A_a \epsilon_+ = \frac{1}{d}\gamma_a \nabla\!\!\!/^A \epsilon_+$

4d	forms	constraints on geometry
Lorentzian	z, ω , $z^2 = 0$	M_4 has a null conformal Killing vector z $\nabla_\mu z_\nu + \nabla_\nu z_\mu = \lambda g_{\mu\nu}$ [cassani,klare,martelli, tomasiello,A.Z.]
Euclidean	j, ω	M_4 is complex (locally) $d\omega = W \wedge \omega$ [klare,tomasiello,A.Z.;dumitruescu,festuccia,seiberg]

Supersymmetry on Curved spaces

More generally, we may ask when we can put a generic supersymmetric theory on a curved background: coupled it the Poincaré supergravity and set the gravitino variation to zero. [festuccia,seiberg]

A theory with an R-symmetry can be coupled to new minimal supergravity which has two auxiliary fields (a_μ, v_μ) with $d(*v) = 0$. The gravitino variation is:

$$\nabla_m \epsilon_+ = -i \left(\frac{1}{2} v^n \gamma_{nm} + (v - a)_m \right) \epsilon_+$$

Reconstructing the Lagrangian

By freezing the supergravity fields $g_{\mu\nu} a_\mu, v_\mu$ to their background value we reconstruct the Lagrangian for the Field Theory

$$-\frac{1}{2} g_{mn} T^{mn} + \left(a_m - \frac{3}{2} v_m \right) J^m + \bar{\psi}_m \mathcal{J}^m - \frac{1}{2} v^m t_m + O(v^2, a^2)$$

and the field transformations (for an euclidean chiral multiplet $\mathcal{F} = (\phi, \psi_\pm, F)$):

$$\delta\phi = -\overline{\epsilon_+}\psi_+ \ , \qquad \delta\bar{\phi} = 0$$

$$\delta\psi_+ = F\epsilon_+ \ , \qquad \delta\psi_- = -\nabla^a_m \bar{\phi}\gamma^m \epsilon_+$$

$$\delta F = 0 \qquad \delta\bar{F} = \overline{\epsilon_+}\gamma^m \left(\nabla^a_m + \frac{i}{2} v_m \right) \psi_-$$

[festuccia,seiberg]

Supersymmetry on Curved spaces

Conformal Killing spinors are closely related to solutions of the new minimal condition. Defining $\not\nabla^A \epsilon_+ \equiv 2i\not v \epsilon_+$ we can map a CKS to (and viceversa)

$$\nabla^A_a \epsilon_+ - \frac{1}{4}\gamma_a \not\nabla^A \epsilon_+ = 0 \qquad \Longrightarrow \qquad \nabla_m \epsilon_+ = -i\left(\frac{1}{2}v^n\gamma_{nm} + (v-a)_m\right)\epsilon_+$$

Condition for coupling a supersymmetric theory to new minimal supergravity $(g_{\mu\nu}, a_\mu, v_\mu)$ same as the condition for existence of a CKS with $a \equiv A + \frac{3}{2}v$

- One Euclidean supercharge ϵ_+: M_4 should be a complex manifold

  [klare,tomasiello,A.Z.;dumitruescu,festuccia,seiberg]

- One Lorentzian supercharge: M_4 should have a null Killing Vector z

  [cassani,klare,martelli, tomasiello,A.Z.]

Poincaré supergravity and its auxiliary fields arise from the gauge fixing of the superconformal algebra using compensators.

Open Problems and Conclusions

- Physics-wise: exact results for quantum path integrals in curved space
- What are the conditions in other dimensions and other super symmetries? interesting for (2,0),...
- Interesting connection to holography and the AdS/CFT correspondence

An example: 3d susy on spheres

The path integral localizes on very simple configurations with $A_\mu = 0$ and a constant σ and can be reduced to a matrix model

[Kapustin,Willet,Yakoov]

$$\begin{aligned} \delta A_\mu &= -\frac{i}{2}\lambda^\dagger \gamma_\mu \varepsilon \\ \delta\sigma &= -\frac{1}{2}\lambda^\dagger \varepsilon \\ \delta\lambda &= \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu \partial_\mu \sigma - \frac{1}{R}\sigma\right)\varepsilon \end{aligned}$$

$$\nabla_\mu \varepsilon = \frac{i}{2}\gamma_\mu \varepsilon$$

For ABJM, for example:

$$\int \prod_{i=1}^{N_1} du_i \prod_{j=1}^{N_2} dv_j \frac{\prod_{i<j} \sinh^2 \frac{u_i - u_j}{2} \sinh^2 \frac{v_i - v_j}{2}}{\prod_{i<j} \cosh^2 \frac{u_i - v_j}{2}} e^{\frac{ik}{4\pi}(\sum u_i^2 - \sum v_j^2)}$$

- the partition function scales as $\sim N^{3/2}$ for $N \gg k$ [Drukker,Marino, Putrov]

When we can have susy in 3d curved space?

We can also consider the dimensionally reduced new minimal equation:

$$\nabla_m \chi = -i\,(v^n \sigma_{nm} + (v-a)_m)\,\chi + \frac{v_4}{2}\sigma_m \chi\ , \qquad m,n = 1,2,3$$

- The condition of supersymmetry can be given in terms of a set of vierbein $e_3, o = e_1 + ie_2$ by:

$$\begin{aligned} de_3 &= -(dB + 2\,\mathrm{Im}a) \wedge e_3 + 4 * \mathrm{Re}v + i\,\mathrm{Im}v_4\, o \wedge \bar{o} \\ do &= (2\,v_4 e_3 + 2i\,a - dB) \wedge o \\ dB &= 2\,\mathrm{Im}(v-a) + i\,\mathrm{Re}v \llcorner (o \wedge \bar{o}) + \mathrm{Re}v_4 e_3 \end{aligned}$$

- Many examples on spheres, round and squashed.

Digression: 3d susy on spheres

The partition function is supposed to decrease along a RG flow, thus playing the role of a c function in 3D.

- a-theorem recently proved in 4d

  [Komargodski-Schwimmer]

- general arguments for all dimensions for theories with holographic duals

  [Girardello,Petrini,Porrati,A.Z.; Gubser,Freedman,Pilch,Warner; Myers]

- link to entanglement entropy

  [Casini,Huerta,Myers]

Digression: 3d susy on spheres

The partition function can be refined as a function of the conformal dimensions of the fields $F(\Delta_i)$. Δ_i parametrizes the different ways of coupling the theory to curved space on S^3.

$$\delta\phi = 0\,, \qquad \delta\phi^\dagger = \psi^\dagger\varepsilon$$

$$\delta\psi = (-i\not{D}\phi - i\sigma\phi + \frac{\Delta}{r}\phi)\varepsilon\,, \qquad \delta\psi^\dagger = \varepsilon^T F^\dagger$$

$$\delta F = \varepsilon^T(-i\not{D}\psi + i\sigma\psi + \frac{1}{r}(\frac{1}{2} - \Delta)\psi + i\lambda\phi) \qquad \delta F^\dagger = 0$$

R symmetry ambiguity due to conserved global symmetry:

$$J_R = J_R^{(0)} + \sum_i q_i J^i\,, \qquad \Delta = R$$

Turn on a background value for the scalar partner of J^i: $\sigma_i = m_i + i\Delta_i$

$F(\Delta_i)$ is maximized at the exact R-symmetry

[Jafferis; Hama,Hosomici,Lee]

CFTs on Curved spaces

General vacua of a bulk effective action

$$\mathcal{L} = -\frac{1}{2}\mathcal{R} + F_{\mu\nu}F^{\mu\nu} + V\ldots$$

with a metric

$$ds^2_{d+1} = \frac{dr^2}{r^2} + (r^2 ds^2_{M_d} + O(r)) \qquad A = A_{M_d} + O(1/r)$$

and a gauge fields profile, correspond to CFTs on a d-manifold M_d and a non trivial background field for the R-symmetry

$$L_{CFT} + J^\mu A_\mu$$

SCFTs on Curved spaces

Requiring that some supersymmetry is preserved: [...,klare,tomasiello,A.Z.]

$$\left(\nabla_M^A + \frac{1}{2}\gamma_M + \frac{i}{2}\not{F}\gamma_M\right)\epsilon = 0 \qquad \nabla_M^A \equiv \nabla_M - iA_M$$

Near the boundary:

$$\epsilon = r^{\frac{1}{2}}\epsilon_+ + r^{-\frac{1}{2}}\epsilon_-$$

$$(\nabla_a - iA_a)\epsilon_+ + \gamma_a\epsilon_- = 0 \qquad \Longrightarrow \qquad \nabla_a^A\epsilon_+ = \frac{1}{d}\gamma_a\not{\nabla}^A\epsilon_+$$

Existence of a Conformal Killing Spinor.

Back to Holography

The dual of a supersymmetric CFT on M_4 is an asymptotically AdS_5 bulk space

$$ds_5^2 = \frac{dr^2}{r^2} + (r^2 ds_{M_4}^2 + O(r))$$

solving the supersymmetric conditions

$$\left(\nabla_M^A + \frac{1}{2}\gamma_M + \frac{i}{2}\not{F}\gamma_M\right)\epsilon = 0$$

of minimal gauged supergravity in 5d.

Holographic perspective

In the lorentzian case supersymmetric bulk solutions are classified. [gauntlett,guowski]

Supersymmetry requires a Killing vector V in the bulk, null or time-like

- For V time-like,
  the metric must be a time-like fibration over a Kähler manifold.
- For V null,
  the metric is of the form $ds^2 = H^{-1}(dudy + Fdu^2) + H^2 h_{mn} dx^m dx^n$

An Example

For V time-like, supersymmetry requires the metric to be a time-like fibration over a Kähler manifold.

Take just AdS in global coordinates and send $\phi \to \tilde{\phi} - 2t$ ($\tilde{\sigma}_3 = \sigma_3 - 2dt$)

$$-(1+r^2)dt^2+\frac{dr^2}{r^2+1}+\frac{r^2}{4}(\sum_{i=1}^{3}\sigma_i^2) = -(dt+\frac{r^2}{2}\tilde{\sigma}_3)^2+\frac{dr^2}{r^2+1}+\frac{r^2}{4}(\sigma_1^2+\sigma_2^2+(r^2+1)\tilde{\sigma}_3^2)$$

$$V = dt + r^2\tilde{\sigma}_3 = r^2 z + \cdots$$

Holographic perspective

The null or time-like Killing vector V in the bulk reduces to a null conformal killing vector on the boundary

$$V = r^2 z + \ldots$$

One can show that

- the condition of supersymmetry in the bulk reduces asymptotically to those we have found on the boundary and nothing more
- given a boundary metric M_4 with a null conformal Killing vector, a bulk solution that can be determined order by order in $1/r$
- open problem: when the bulk metric is **regular**?

Some examples

There are few explicit supersymmetric examples in five and four dimensions:

- SCFTs on spheres: standard AdS_d
- SCFTs on 3d squashed spheres: evaluating $\mathcal{N}^{3/2}$ free energies [Martelli,Passias,Sparks]
- Some Lorentzian examples [Gauntlett-Gutowski-Suryanarayana]